The J-equation and the deformed Hermitian-Yang-Mills equation

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7/15/2022

Motivation

- Study constant scalar curvature Kähler metrics
- Study special Lagrangian submanifolds on Calabi-Yau manifolds.

Constant scalar curvature Kähler metric

Question (Calabi)

Let (M^n, J, ω_0) *be a Kähler manifold. When does there exist* φ *such that the scalar curvature* R_{ω_φ} *of the metric*

$$\omega_\varphi = \omega_0 + i\partial\bar{\partial}\varphi > 0$$

is a constant?

Remark (Chern-Weil)

If R_{ω_φ} is a constant, then it equals to

$$\underline{R} = \frac{2\pi n c_1(M) \cdot [\omega_0]^{n-1}}{[\omega_0]^n}.$$

Calabi-Yau manifold

Special case:

Theorem (Calabi's conjecture, Yau's theorem)

Let $(M^n, J, \omega_0, \Omega)$ be a Kähler manifold with a holomorphic $(n, 0)$-form. Then there exist φ such that

$$\omega_\varphi^n = c\Omega \wedge \bar{\Omega}$$

for a constant c.

In honor of Calabi and Yau, $(M^n, J, \omega_\varphi, \Omega)$ is called a Calabi-Yau manifold. Its Ricci curvature is zero and therefore, is a special case of constant scalar curvature Kähler metrics. Calabi-Yau metrics play an important role in mathematical physics.

Special Lagrangian

Definition (Harvey-Lawson)

A submanifold L with real dimension n is called a special Lagrangian submanifold of a Calabi-Yau manifold (M^n, J, ω, Ω) if $\omega|_L = 0$ and $\mathrm{Im}(e^{-i\hat{\theta}}\Omega)|_L = 0$ for a constant $\hat{\theta}$.

Deformed Hermitian-Yang-Mills equation

Theorem (Harvey-Lawson)

Let F be a function from $\mathbb{R}^n$ to $\mathbb{R}$. Then $\nabla F : \mathbb{R}^n = \mathrm{Re}\mathbb{C}^n \to \mathbb{R}^n = \mathrm{Im}\mathbb{C}^n$. *The the graph of ∇F is a special Lagragian submanifold in $\mathbb{C}^n$ if and only if*

$$\mathrm{Im}(e^{-i\hat{\theta}} \det(I + i\mathrm{Hess}F)) = 0.$$

Definition

Let (M^n, J, χ) be a Kähler manifold with a Kähler metric χ and a closed (1,1)-form ω_0. Then we say that $\omega_\varphi = \omega_0 + i\partial\bar{\partial}\varphi$ satisfies the deformed Hermitian-Yang-Mills equation if

$$\mathrm{Im}(e^{-i\hat{\theta}}(\chi + i\omega_\varphi)^n) = 0.$$

Mirror symmetry

By Mariño-Minasian-Moore-Strominger and Leung-Yau-Zaslow, the dHYM equation on a Calabi-Yau manifold is mirror to the special Lagrangian equation on the mirror Calabi-Yau manifold.

Type IIB string theory	Type IIA string theory
Complex	Symplectic
Coherent sheaves	Lagrangian submanifolds
dHYM	Special Lagragian

Limit of dHYM equation

In local coordinates,

$$\chi = \sqrt{-1}\sum_{i=1}^{n} dz^i \wedge d\bar{z}^i, \omega_\varphi = \sqrt{-1}\sum_{i=1}^{n} \lambda_i dz^i \wedge d\bar{z}^i,$$

then the dHYM equation is the same as

$$\sum_{i=1}^{n} \arctan(\lambda_i) = \hat{\theta} \mod 2\pi.$$

In "small radius limit",

$$\lim_{t\to\infty} \frac{\frac{\pi}{2} - \sum_{i=1}^{n} \arctan(t\lambda_i)}{t} = \sum_{i=1}^{n} \frac{1}{\lambda_i}.$$

$$\sum_{i=1}^{n} \frac{1}{\lambda_i} = c$$

is called the J-equation.

Limit of dHYM equation

In "large radius limit",

$$\lim_{t\to 0}\frac{\sum_{i=1}^{n}\arctan(t\lambda_i)}{t}=\sum_{i=1}^{n}\lambda_i.$$

When ω_φ is the curvature F of a line bundle,

$$\sum_{i=1}^{n}\lambda_i=\mathrm{tr}_\chi F=c$$

is the Hermitian-Yang-Mills equation.

Theorem (Donaldson, Uhlenbeck-Yau)

The Hermitian-Yang-Mills equation is solvable if and only if the vector bundle is stable.

Question

How about the deformed Hermitian-Yang-Mills equation and the J-equation?

Constant scalar curvature Kähler metric

- Calabi: Let (M^n, J, ω_0) be a Kähler manifold. When does there exist a Kähler form $\omega_\varphi \in [\omega_0]$ such that the scalar curvature R_{ω_φ} is a constant?
- Special case: Kähler-Einstein.
- Yau: When $c_1(M) = 0$, there always exists a Kähler Ricci-flat metric in $[\omega_0]$.
- Aubin, Yau: When $c_1(M) = -[\omega_0] < 0$, there always exists a Kähler-Einstein metric in $[\omega_0]$.
- Matsushima, Futaki: When $c_1(M) = [\omega_0] > 0$, there may not exist any Kähler-Einstein metric in $[\omega_0]$.
- Yau's conjecture (Motivated by Donaldson-Uhlenbeck-Yau theorem): When $c_1(M) = [\omega_0] > 0$, there exists a Kähler-Einstein metric in $[\omega_0]$ if and only if it is stable.

Variational approach to cscK problem

Theorem (Mabuchi)

Let (M^n, J, ω_0) *be a Kähler manifold. Then there exists a convex functional*

$$K : \mathcal{H} = \{\varphi : \omega_\varphi = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi > 0\} \to \mathbb{R}$$

such that

$$\delta K = \int_M (\underline{R} - R_{\omega_\varphi})\delta\varphi \frac{\omega_\varphi^n}{n!}.$$

The critical points of the K-energy functional are constant scalar curvature metrics

$$R_{\omega_\varphi} = \underline{R} = \frac{2\pi n c_1(M) \cdot [\omega_0]^{n-1}}{[\omega_0]^n}.$$

Variational approach to cscK problem

Theorem (Chen)

$$K(\varphi) = \int_M \log(\frac{\omega_\varphi^n}{\omega_0^n})\omega_\varphi^n + \mathcal{J}_{-Ric(\omega_0)}(\varphi).$$

In general, if M^n is a Kähler manifold and χ is a closed (1,1)-form, then

$$\mathcal{J}_\chi(\varphi) = \frac{1}{n!}\int_M \varphi \sum_{k=0}^{n-1} \chi\wedge\omega_0^k\wedge\omega_\varphi^{n-1-k} - \frac{c_0}{(n+1)!}\int_M \varphi \sum_{k=0}^{n} \omega_0^k\wedge\omega_\varphi^{n-k},$$

and c_0 is the constant given by

$$\int_M \chi \wedge \frac{\omega_0^{n-1}}{(n-1)!} - c_0\frac{\omega_0^n}{n!} = 0.$$

$\mathcal{J}_\chi(\varphi)$ is also convex if $\chi > 0$.

J-equation

Proposition (Chen)

The φ is the critical point for $\mathcal{J}_\chi$ if and only if it satisfies the J-equation

$$\mathrm{tr}_{\omega_\varphi}\chi = \sum_{i=1}^{n} \frac{1}{\lambda_i} = c_0$$

We get the J-equation from the cscK problem!
Even though this talk focus on the study of the J-equation and the dHYM using ideas from cscK equations, I would like to remark that the J-equation first appeared in Donaldson's study of symplectic geometry and Chen's study of cscK equation was the second appearance.

Coerciveness and uniform geodesic stability

Question

When does a convex functional has a critical point?

Lemma (Coerciveness)

A smooth strictly convex function F on $\mathbb{R}^n$ has a critical point if and only if there exist constants $\epsilon > 0$ and C such that

$$F(x) \geq \epsilon|x| + C.$$

Lemma (Uniform geodesic stability)

A smooth Lipschitz strictly convex function F on $\mathbb{R}^n$ has a critical point if and only if there exist constants $\epsilon > 0$ such that

$$\lim_{t\to\infty} \frac{F(x+\alpha t)}{t} \geq \epsilon \lim_{t\to\infty} \frac{|x+\alpha t|}{t} = \epsilon|\alpha|$$

for all geodesic $x + \alpha t$ on $\mathbb{R}^n$.

Geodesic stability and stability

Lemma (Uniform geodesic stability)

A smooth Lipschitz strictly convex function F on $\mathbb{R}^n$ has a critical point if and only if there exist constants $\epsilon > 0$ such that

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for all geodesic $x + \alpha t$ on $\mathbb{R}^n$ when $\alpha \in \mathbb{Q}^n$.

Coerciveness

On the space of Kähler potentials $\mathcal{H}$, $\mathrm{tr}_{\omega_\varphi}\omega_0 = n$ is trivially solvable, so $J_{\omega_0}(\varphi)$ plays the role of $\sqrt{|x|^2+1}$ on $\mathbb{R}^n$. Strict convexity means no holomorphic vector fields.

Definition

A functional F on $\mathcal{H}$ is called a coercive functional if there exist constants ϵ and C such that $F(\varphi) \geq \epsilon J_{\omega_0}(\varphi) + C$.

Theorem (Tian $\Rightarrow$, Ding-Tian $\Leftarrow$)

Existence of a Fano Kähler-Einstein metric $\Longleftrightarrow$ *Coerciveness of K-energy.*

Theorem (Darvas-Rubinstein $\Rightarrow$, Chen-Cheng $\Leftarrow$)

Existence of a constant scalar curvature Kähler metric $\Longleftrightarrow$ *Coerciveness of K-energy.*

Coerciveness and uniform stability

Theorem (Collins-Székelyhidi)

Let χ, ω_0 be Kähler forms on M, then
$\mathrm{tr}_{\omega_\varphi}\chi = c_0$ *is solvable* $\iff$ J_χ *is coercive.*

Trouble: It's very hard to verify coerciveness, sometimes as hard as solving the equation directly!

Theorem (Coerciveness $\Rightarrow$ Uniformly geodesic stability is trivial, Chen-Cheng $\Leftarrow$)

Existence of a constant scalar curvature Kähler metric $\iff$ *geodesic stability.*

Idea of Chen-Cheng's result: Twist the cscK equation $R(\omega_\varphi) = \underline{R}$ with the J-equation $\mathrm{tr}_{\omega_\varphi}\chi = c_0$ to get the twisted cscK equation

$$-tR(\omega_\varphi) + (1-t)\mathrm{tr}_{\omega_\varphi}\chi = -t\underline{R} + (1-t)c_0.$$

Uniform stability

Lemma (Uniform stability)

A smooth Lipschitz strictly convex function F on $\mathbb{R}^n$ has a critical point if and only if there exist constants $\epsilon > 0$ such that

$$\lim_{t\to\infty} \frac{F(x+\alpha t)}{t} \geq \epsilon \lim_{t\to\infty} \frac{|x+\alpha t|}{t} = \epsilon|\alpha|$$

for all geodesic $x + \alpha t$ on $\mathbb{R}^n$ when $\alpha \in \mathbb{Q}^n$.

It is still hard to verify geodesic stability because the it's hard to write down ALL geodesics explicitly. So we want a analogy of the "rational slope geodesic" in $\mathbb{R}^n$. They are called test configuration. The slope $\lim_{t\to\infty} \frac{K(\varphi_t)}{t}$ is called the Donaldson-Futaki invariant.

Coerciveness and uniform stability

Definition (Yau,Tian,Donaldson,Sjöstrom Dyrefelt,Dervan-Ross)

$(M, J, [\omega_0])$ is called uniformly K-stable if there exists $\epsilon > 0$ such that for any test configuration $\mathcal{X}$, the Donaldson-Futaki invariant $DF(\mathcal{X})$ is larger than ϵ times the slope for J_{ω_0}.

Theorem (Chen-Donaldson-Sun)

K-stability $\iff$ *the existence of Fano Kähler-Einstein metric.*

Definition (Ross-Thomas)

Any subvariety V of M induces a test configuration. The uniform stability among those geodesics is called the uniform slope stability.

Same definition holds for J_χ functional.

Uniform stability and uniform slope stability

Not known whether uniform stability is enough for cscK problem. As for uniform slope stability, it's not even known for Fano Kähler-Einstein metrics.
People still hope to solve this problem using twisted cscK equation. Not known even for J-equation before my work.

Main theorem

Theorem (C.)

The followings are equivalent:

1. *The J-equation* $\mathrm{tr}_{\omega_\varphi}\chi = c$ *is solvable.*
2. *The* J_χ *functional is coercive.*
3. $[\omega_0]$ *is uniformly* J_χ *geodesic stable.*
4. $[\omega_0]$ *is uniformly* J_χ *stable for all test configurations.*
5. $[\omega_0]$ *is uniformly* J_χ *slope stable for all subvarieties.*
6. $\int_V c_0\omega_0^p - p\chi \wedge \omega_0^{p-1} \geq \epsilon(n-p)\int_V \omega_0^p.$

(1) $\iff$ (2) is due to Collins-Székelyhidi. (2) $\Rightarrow$ (3) is the definition for geodesic stability. (3) $\Rightarrow$ (4) is the definition of stability for all test configurations by Yau, Tian, Donaldson, Sjöstrom Dyrefelt, Dervan-Ross. (4) $\Rightarrow$ (5) is the definition of slope stability similar to Ross-Thomas. (5) $\Rightarrow$ (6) is due to Lejmi-Székelyhidi. Only (6) $\Rightarrow$ (1) is new.

dHYM equation

The condition $c_0\omega_0^p - p\chi \wedge \omega_0^{p-1} \geq \epsilon(n-p)\omega_0^p$ is equivalent to $\frac{1}{\lambda_{i_1}} + ...\frac{1}{\lambda_{i_p}} \leq c_0 - (n-p)\epsilon$. My main contribution is that

$$\int_V c_0\omega_0^p - p\chi \wedge \omega_0^{p-1} \geq \epsilon(n-p)\int_V \omega_0^p$$

implies the solvability of the J-equation.
For deformed Hermitian-Yang-Mills equation $\sum_{i=1}^n(\frac{\pi}{2} - \arctan\lambda_i) = \hat{\theta}$, the corresponding inequality is

$$\frac{\pi}{2} - \arctan\lambda_{i_1} + ... + \frac{\pi}{2} - \arctan\lambda_{i_p} \leq \hat{\theta} - (n-p)\epsilon.$$

When $0 < \hat{\theta} < \pi$ (supercritical case), I proved that a similar uniform inequality about integrals on subvarieties implies the solvability of the dHYM equation. The actual statement of the theorem involves the technical issue of dealing with $\mod 2\pi$ and is too complicated to discuss here, see [Chen, The J-equation and the supercritical deformed Hermitian-Yang-Mills equation, Inv. Math. (2021)] for more details.

Generalized Monge-Ampère equation

Conjecture (Lemji-Székelyhidi)

The J-equation $c\omega_\varphi^n = n\chi \wedge \omega_\varphi^{n-1}$ *is solvable if and only if* $\int_V c\omega_0^p - p\chi \wedge \omega_0^{p-1} > 0$ *for all* $p < n$.

Conjecture (Székelyhidi)

The generalized Monge-Ampère equation $\omega_\varphi^n = \sum_{k=0}^{n-1} c_k \chi^{n-k} \wedge \omega_\varphi^k$ *is solvable if and only if* $\int_V \omega_0^p - \sum_{k=n-p}^{n-1} \frac{c_k k!}{n!} \frac{p!}{(k+p-n)!} \chi^{n-k} \wedge \omega_0^{k+p-n} > 0$ *for all* $p < n$.

Datar-Pingali generalized my method and solved Székelyhidi's conjecture on projective manifolds in July 2020. Song generalized my method and solved Lemji-Székelyhidi's conjecture for the J-equation without the projective assumption in December 2020.

Idea of proof

By a generalization of Song-Weinkove to reduce the problem to the search of ω_φ such that $\frac{1}{\lambda_{i_1}} + ... \frac{1}{\lambda_{i_{n-1}}} < c_0$ for all $i_1 < ... < i_{n-1}$.
Use the continuity method: Solve $\mathrm{tr}_{\omega_t}\chi + c_t\frac{\chi^n}{\omega_t^n} = c_0$ for $\omega_t \in [t\omega_0]$. (This path originates in the solution of Lemji-Székelyhidi's conjecture in the toric case by Collins-Székelyhidi.) It's solvable for large t and it's open by the first step. To prove the closeness, we trivially get a current satisfying the equation. We need a smoothing.

Błocki and Kołodziej's smoothing method

On $\mathbb{C}^n$, a current can be smoothed by convolution with the modifier. We need to glue the local potential functions. Consider $-\sqrt{-1}\sum_{i=1}^{n} dz_i \wedge d\bar{z}_i$ on the torus $T^{2n} = \mathbb{C}^n/\mathbb{Z}^{2n}$. Then we choose finitely many points p_j on T^{2n}. The local potential near p_j is $-|z-p_j|^2$. Then

$$\sqrt{-1}\partial\bar{\partial}\max\{-|z-p_1|^2, -|z-p_2|^2, ...\}$$

is in the zero class. The main benefit of the shift from $[-\sqrt{-1}dz_i \wedge d\bar{z}_i]$ to the zero class is that $\max\{-|z-p_1|^2, -|z-p_2|^2, ...\}$ cannot be affected by the function $-|z-p_j|^2$ away from p_j so that we do not need to worry about the fact that $-|z-p_j|^2$ can only be defined locally. So if we are allowed to change the Kähler class a little bit, then we can glue the local potentials as long as they are close to each other.

Błocki and Kołodziej's smoothing method

The difference between the local potentials is small if the Lelong number $\nu(x)$ is small.

Theorem (Siu)

For any $\epsilon > 0$, $\{x : \nu(x) \geq \epsilon\}$ is a subvariety.

This is how the stability conditions on subvarieties come in! We still need to figure out how to adjust the Kähler class a little bit.

Demailly-Paun's theorem

Theorem (Demailly-Paun)

Let X be a compact Kähler manifold. Then the Kähler cone K of X is one of the connected components of the set P of real (1,1)-cohomology classes $[\alpha]$ which are numerically positive on analytic cycles, i.e. such that $\int_Y \alpha^p > 0$ for every irreducible analytic set Y in X, $p = \dim Y$.

Proposition (Demailly-Paun)

Any nef and big class contains a Kähler current.

Demailly-Paun's idea

Proposition (Demailly-Paun)

Any nef and big class contains a Kähler current.

Definition

If χ is a Kähler class and α is a closed (1,1)-form on M. Then $[\alpha]$ is called nef and big if $\int_M \alpha^n > 0$ and there exists a smooth Kähler form in $[\alpha + t\chi]$ for all $t > 0$. α_φ is called a Kähler current if there exists $\epsilon > 0$ such that $\alpha - \epsilon\chi$ is a positive current.

Idea: Use Yau's solution to the Calabi conjecture to get $(\alpha + t\chi + i\partial\bar{\partial}\varphi_t)^n = f\chi^n$. By choosing f properly, we can concentrate the mass of $\alpha + t\chi + i\partial\bar{\partial}\varphi_t$. When we concentrate the mass on the diagonal inside $M \times M$, then the push down of it is a Kähler current.

Sketch of the proof

- I use Demailly-Paun's idea to get the extra $\epsilon\chi$ to apply Błocki and Kołodziej's smoothing method to get a smooth subsolution. Then a minor generalization of Song-Weinkove's method provides the smooth solution. This is the idea to solve the J-equation.
- For the dHYM equation in the supercritical case, we get the solution from a subsolution as a minor generalization of Székelyhidi's method (Székelyhidi's method is a generalization of Caffarelli-Nirenberg-Spruck's estimate) and also Collins-Jacob-Yau's method. A technical observation that $\tan(\frac{n\pi}{2} - \sum_i \arctan(\lambda_i))$ is convex is due to Takahashi.

Open problems

Question

How about the special Lagrangian equation?

Question

How about the twisted cscK equation and the cscK equation?

Question

How about coassociative submanifolds on a G_2-manifold?

The product of a special Lagrangian submanifold with S^1 is a coassociative submanifold.

Question

How about the deformed G_2-instanton equation?

The product of a solution of the dHYM equation with S^1 is a solution of the deformed G_2-instanton equation.

Thanks

Thank you for your attention!